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“Reducible second-class theories”

–Summary of Ph.D. thesis–

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1 Reducible second-class theories

The main subject approached in the thesis is the problem of the irreducible analysis of the second-class constraints reducible of an arbitrary order. The approach in irreducible manner of the reducible second-class theories is based on the following steps: i) we express the Dirac bracket for the reducible system in terms of an invertible matrix; ii) we construct an irreducible second-class system (on a larger phase-space) equivalent to the original reducible one; iii) we derive of the Dirac bracket with respect to the irreducible second-class constraints; iv) we prove the fact that the fundamental Dirac brackets derived within the irreducible and original reducible settings coincide (weakly); v) the application of the general procedure on various models. We initially approach second-class constraints reducible of order two and three by implementing the main steps mentioned above, and then generalize these results to an arbitrary order of reducibility.

1.1 The irreducible approach to second-order reducible second-class constraints

1.1.1 Second-order reducible second-class constraints

We start with a system locally described by N canonical pairs $z^a = (q^i, p_i)$, subject to some constraints

$$\chi_{\alpha_0}\left(z^a\right) \approx 0,\ \alpha_0 = \overline{1, M_0}. \tag{1}$$

In addition, we presume that the functions χ_{α_0} are not all independent, but there exist some nonvanishing functions $Z^{\alpha_0}_{\alpha_1}$ and $Z^{\alpha_1}_{\alpha_2}$ such that

$$Z^{\alpha_0}_{\alpha_1}\chi_{\alpha_0} = 0,\ \alpha_1 = \overline{1, M_1}, \tag{2}$$

$$Z^{\alpha_1}_{\alpha_2}Z^{\alpha_0}_{\alpha_1} \approx 0,\ \alpha_2 = \overline{1, M_2}. \tag{3}$$

We will assume that the reducibility stops at order two, so the functions $Z^{\alpha_1}_{\alpha_2}$ are by hypothesis taken to be independent.

The constraints (1) are purely second class if any maximal, independent set of $M \equiv M_0 - M_1 + M_2$ constraint functions χ_A $(A = 1, \cdots, M)$ among χ_{α_0} is such that the matrix

$$C^{(2)}_{AB} = \left[\chi_A, \chi_B\right], \tag{4}$$

is invertible.

In terms of independent constraints, the Dirac bracket takes the form

$$\left[F, G\right]^{(2)*} = \left[F, G\right] - \left[F, \chi_A\right] M^{(2)AB} \left[\chi_B, G\right], \tag{5}$$

where $M^{(2)AB} C^{(2)}_{BC} \approx \delta^A_C$.

We can rewrite the Dirac bracket (5) without finding a definite subset of independent second-class constraints as follows. We start with the matrix

$$C^{(2)}_{\alpha_0\beta_0} = [\chi_{\alpha_0}, \chi_{\beta_0}], \tag{6}$$

which clearly is not invertible because

$$Z_{\alpha_1}^{\alpha_0} C^{(2)}_{\alpha_0\beta_0} \approx 0. \tag{7}$$

Let $\bar{A}_{\alpha_0}^{\alpha_1}$ be some functions chosen such that satisfy the condition

$$rang\left(Z_{\alpha_1}^{\alpha_0} \bar{A}_{\alpha_0}^{\beta_1}\right) \equiv rang\left(D_{\alpha_1}^{\beta_1}\right) = M_1 - M_2. \tag{8}$$

We introduce an antisymmetric matrix $M^{(2)\alpha_0\beta_0}$ through the relation

$$C^{(2)}_{\alpha_0\gamma_0} M^{(2)\gamma_0\beta_0} \approx D_{\alpha_0}^{\beta_0} \equiv \delta_{\alpha_0}^{\beta_0} - \bar{A}_{\alpha_0}^{\beta_1} Z_{\beta_1}^{\beta_0}, \tag{9}$$

such that the formula

$$[F, G]^{(2)*} = [F, G] - [F, \chi_{\alpha_0}]\, M^{(2)\alpha_0\beta_0} [\chi_{\beta_0}, G], \tag{10}$$

defines the same Dirac bracket like (5) on the surface (1).

It can be proved that for systems with second-stage reducible second-class constraints the Dirac bracket can be written in terms of an invertible matrix.

Teorema 1 *There exists an invertible, antisymmetric matrix* $\mu^{(2)\alpha_0\beta_0}$ *, in terms of which the Dirac bracket (10) becomes*

$$[F, G]^{(2)*} = [F, G] - [F, \chi_{\alpha_0}]\, \mu^{(2)\alpha_0\beta_0} [\chi_{\beta_0}, G]. \tag{11}$$

on the surface (1).

The relationship between the invertible matrix $\mu^{(2)\alpha_0\beta_0}$ and the matrix $M^{(2)\alpha_0\beta_0}$ is given by a relation

$$M^{(2)\alpha_0\beta_0} \approx D^{\alpha_0}_{\lambda_0} \mu^{(2)\lambda_0\sigma_0} D^{\beta_0}_{\sigma_0}. \tag{12}$$

1.1.2 Intermediate system

We introduce some new variables, $(y_{\alpha_1})_{\alpha_1=1,\cdots,M_1}$ with the Poisson brackets

$$[y_{\alpha_1}, y_{\beta_1}] = \omega_{\alpha_1\beta_1}, \tag{13}$$

and consider the system subject to the reducible second-class constraints

$$\chi_{\alpha_0} \approx 0,\ y_{\alpha_1} \approx 0. \tag{14}$$

The Dirac bracket on the phase-space locally parameterized by the variables $(z^a,\ y_{\alpha_1})$, corresponding to the above second-class constraints reads as

$$\begin{aligned} [F, G]^{(2)*}\Big|_{z,y} &= [F, G] - [F, \chi_{\alpha_0}]\, \mu^{(2)\alpha_0\beta_0} [\chi_{\beta_0}, G] \\ &\quad - [F, y_{\alpha_1}]\, \omega^{\alpha_1\beta_1} [y_{\beta_1}, G]. \end{aligned} \tag{15}$$

The Dirac bracket (15) coincide (weakly) with that written in terms of invertible matrix $\mu^{(2)\alpha_0\beta_0}$

$$[F, G]^{(2)*}\Big|_{z,y} \approx [F, G]^{(2)*}. \tag{16}$$

1.1.3 Irreducible system

Teorema 2 *There exists a set of constraints (on the larger phase-space (z^a, y_{α_1}))*

$$\tilde{\chi}_{\alpha_0} = \chi_{\alpha_0} + A^{\alpha_1}_{\alpha_0} y_{\alpha_1} \approx 0,\ \tilde{\chi}_{\alpha_2} = Z^{\alpha_1}_{\alpha_2} y_{\alpha_1} \approx 0, \tag{17}$$

such that:

(i)

$$\tilde{\chi}_{\alpha_0} \approx 0,\ \tilde{\chi}_{\alpha_2} \approx 0 \Leftrightarrow \chi_{\alpha_0} \approx 0,\ y_{\alpha_1} \approx 0. \tag{18}$$

(ii)define an irreducible set of second-class constraints, i.e. the matrix

$$C_{\Delta\Delta'} = [\tilde{\chi}_\Delta, \tilde{\chi}_{\Delta'}], \tag{19}$$

is invertible, where $\tilde{\chi}_\Delta = (\tilde{\chi}_{\alpha_0}, \tilde{\chi}_{\alpha_2})$.

The functions $A^{\alpha_1}_{\alpha_0}$ are defined by the relation

$$\bar{A}^{\alpha_1}_{\alpha_0} = A^{\beta_1}_{\alpha_0} \hat{e}^{\alpha_1}_{\beta_1}, \tag{20}$$

where $\hat{e}^{\alpha_1}_{\beta_1}$ are the elements of an invertible matrix.

The Dirac bracket associated with the irreducible second-class constraints (17) takes the concrete form

$$\begin{aligned}
\left.[F,G]^{(2)*}\right|_{\text{ired}} = {} & [F,G] - [F,\tilde{\chi}_{\alpha_0}]\, \mu^{(2)\alpha_0\beta_0} [\tilde{\chi}_{\beta_0}, G] - \\
& [F,\tilde{\chi}_{\alpha_0}]\, Z^{\alpha_0}_{\gamma_1} \hat{e}^{\gamma_1}_{\sigma_1} \omega^{\sigma_1\lambda_1} A^{\tau_2}_{\lambda_1} \bar{D}^{\beta_2}_{\tau_2} [\tilde{\chi}_{\beta_2}, G] - \\
& [F,\tilde{\chi}_{\alpha_2}]\, \bar{D}^{\alpha_2}_{\lambda_2} A^{\lambda_2}_{\sigma_1} \omega^{\sigma_1\lambda_1} \hat{e}^{\gamma_1}_{\lambda_1} Z^{\beta_0}_{\gamma_1} [\tilde{\chi}_{\beta_0}, G] - \\
& [F,\tilde{\chi}_{\alpha_2}]\, \bar{D}^{\alpha_2}_{\lambda_2} A^{\lambda_2}_{\sigma_1} \omega^{\sigma_1\lambda_1} A^{\tau_2}_{\lambda_1} \bar{D}^{\beta_2}_{\tau_2} [\tilde{\chi}_{\beta_2}, G].
\end{aligned} \tag{21}$$

Teorema 3 *The Dirac bracket with respect to the irreducible second-class constraints coincides with that of the intermediate system*

$$\left.[F,G]^{(2)*}\right|_{\text{ired}} \approx \left.[F,G]^{(2)*}\right|_{z,y}. \tag{22}$$

Combining (16) and (22), we reach the result

$$[F,G]^{(2)*} \approx \left.[F,G]^{(2)*}\right|_{\text{ired}}. \tag{23}$$

1.2 Generalization to an arbitrary reducibility order L

1.2.1 Reducible second-class constraints of order L

We will consider the case of a system of second-class constraints, reducible of an arbitrary order L

$$Z^{\alpha_0}_{\alpha_1} \chi_{\alpha_0} = 0, \qquad Z^{\alpha_1}_{\alpha_2} Z^{\alpha_0}_{\alpha_1} \approx 0, \ldots, \qquad Z^{\alpha_{L-1}}_{\alpha_L} Z^{\alpha_{L-2}}_{\alpha_{L-1}} \approx 0, \tag{24}$$

with $\alpha_k = \overline{1, M_k}$ for each $k = \overline{1, L}$. In addition, the reducibility functions of maximum order (L), $Z^{\alpha_{L-1}}_{\alpha_L}$, are assumed to be all independent. Consequently, the number of independent second-class constraints is equal to $M \equiv \sum\limits_{k=0}^{L} (-)^k M_k$.

The Dirac bracket in terms of M independent functions χ_A takes the form

$$[F, G]^{(L)*} = [F, G] - [F, \chi_A] M^{(L)AB} [\chi_B, G], \qquad A = \overline{1, M}, \tag{25}$$

where $C^{(L)}_{AB} M^{(L)BC} \approx \delta^C_A$, with $C^{(L)}_{AB} = [\chi_A, \chi_B]$.

The matrix of the Poisson brackets among the constraint functions

$$C^{(L)}_{\alpha_0\beta_0} = [\chi_{\alpha_0}, \chi_{\beta_0}] \tag{26}$$

is not invertible due to the relations $Z^{\alpha_0}_{\alpha_1} C^{(L)}_{\alpha_0\beta_0} \approx 0$ but its rank is equal to M.

Let $\left(\bar{A}^{\alpha_k}_{\alpha_{k-1}}\right)_{k=\overline{1,L}}$ be subject to the relations

$$\operatorname{rang}\left(Z^{\beta_{k-1}}_{\alpha_k} \bar{A}^{\gamma_k}_{\beta_{k-1}}\right) \equiv \operatorname{rang}\left(D^{\gamma_k}_{\alpha_k}\right) \approx \sum_{i=k}^{L} (-)^{k+i} M_i, \tag{27}$$

$$\bar{A}^{\alpha_{k-1}}_{\alpha_{k-2}} \bar{A}^{\alpha_k}_{\alpha_{k-1}} \approx 0. \tag{28}$$

We introduce an antisymmetric matrix, of elements $M^{(L)\alpha_0\beta_0}$, through the relation

$$C^{(L)}_{\alpha_0\beta_0} M^{(L)\beta_0\gamma_0} \approx D^{\gamma_0}_{\alpha_0} \equiv \delta_{\alpha_0}^{\ \beta_0} - \bar{A}_{\alpha_0}^{\ \beta_1} Z_{\beta_1}^{\ \beta_0}, \tag{29}$$

such that

$$[F, G]^{(L)*} = [F, G] - [F, \chi_{\alpha_0}] M^{(L)\alpha_0\beta_0} [\chi_{\beta_0}, G] \tag{30}$$

defines the same Dirac bracket like (25) on the surface (1).

The Dirac bracket for L-order reducible constraints can be expressed in terms of a noninvertible matrix.

Teorema 4 *There exists an invertible, antisymmetric matrix $\mu^{(L)\alpha_0\beta_0}$ such that Dirac bracket (30) takes the form*

$$[F, G]^{(L)*} = [F, G] - [F, \chi_{\alpha_0}] \mu^{(L)\alpha_0\beta_0} [\chi_{\beta_0}, G], \tag{31}$$

on the surface (1).

The relationship between the invertible matrix $M^{(L)\alpha_0\beta_0}$ and the matrix $\mu^{(L)\alpha_0\beta_0}$ is given by the relation

$$M^{(L)\alpha_0\beta_0} \approx D^{\alpha_0}_{\lambda_0} \mu^{(L)\lambda_0\sigma_0} D^{\beta_0}_{\sigma_0}. \tag{32}$$

1.2.2 Intermediate system

We introduce some new variables, $\left(y_{\alpha_{2k+1}}\right)_{\alpha_{2k+1}=\overline{1,M_{2k+1}}}$, with $k=\overline{0,\left[\frac{L-1}{2}\right]}$, exhibiting the Poisson brackets

$$\left[y_{\alpha_i},y_{\beta_j}\right]=\omega_{\alpha_i\beta_j}\delta_{ij}, \tag{33}$$

and consider the system subject to the reducible second-class constraints

$$\chi_{\alpha_0}\approx 0,\qquad \left(y_{\alpha_{2k+1}}\right)_{k=\overline{0,\left[\frac{L-1}{2}\right]}}\approx 0. \tag{34}$$

The Dirac bracket on the phase-space locally parameterized by the variables $\left(z^a,\left(y_{\alpha_{2k+1}}\right)_{k=\overline{0,\left[\frac{L-1}{2}\right]}}\right)$ constructed with respect to the above second-class constraints, reads as

$$\begin{aligned}\left.[F,G]^{(L)*}\right|_{z,y}&=[F,G]-[F,\chi_{\alpha_0}]\,\mu^{(L)\alpha_0\beta_0}\,[\chi_{\beta_0},G]\\&-\sum_{k=0}^{\left[\frac{L-1}{2}\right]}\left[F,y_{\alpha_{2k+1}}\right]\omega^{\alpha_{2k+1}\beta_{2k+1}}\left[y_{\beta_{2k+1}},G\right],\end{aligned} \tag{35}$$

and coincide (weakly) with Dirac bracket written in terms of invertible matrix $\mu^{(L)\alpha_0\beta_0}$

$$\left.[F,G]^{(L)*}\right|_{z,y}\approx[F,G]^{(L)*}. \tag{36}$$

1.2.3 Irreducible system

Teorema 5 *There exists a set of constraints (on the larger phase-space, locally parameterized by* $\left(z^a,\left(y_{\alpha_{2k+1}}\right)_{k=\overline{0,\left[\frac{L-1}{2}\right]}}\right)$*)*

-if L odd

$$\tilde{\chi}_{\alpha_0}\equiv\chi_{\alpha_0}+A_{\alpha_0}^{\alpha_1}y_{\alpha_1}\approx 0, \tag{37}$$

$$\tilde{\chi}_{\alpha_{2k}}\equiv Z_{\alpha_{2k}}^{\alpha_{2k-1}}y_{\alpha_{2k-1}}+A_{\alpha_{2k}}^{\alpha_{2k+1}}y_{\alpha_{2k+1}}\approx 0,\quad k=\overline{1,\left[\frac{L}{2}\right]}; \tag{38}$$

-if L even

$$\tilde{\chi}_{\alpha_0}\equiv\chi_{\alpha_0}+A_{\alpha_0}^{\alpha_1}y_{\alpha_1}\approx 0, \tag{39}$$

$$\tilde{\chi}_{\alpha_{2k}}\equiv Z_{\alpha_{2k}}^{\alpha_{2k-1}}y_{\alpha_{2k-1}}+A_{\alpha_{2k}}^{\alpha_{2k+1}}y_{\alpha_{2k+1}}\approx 0,\quad k=\overline{1,\frac{L}{2}-1}, \tag{40}$$

$$\tilde{\chi}_{\alpha_L}\equiv Z_{\alpha_L}^{\alpha_{L-1}}y_{\alpha_{L-1}}\approx 0; \tag{41}$$

with the following properties:

(i)

$$\left(\tilde{\chi}_{\alpha_{2k}}\right)_{k=\overline{0,\left[\frac{L}{2}\right]}}\approx 0\Leftrightarrow\left(\chi_{\alpha_0}\approx 0,\left(y_{\alpha_{2k+1}}\right)_{k=\overline{0,\left[\frac{L-1}{2}\right]}}\approx 0\right); \tag{42}$$

(ii) define an irreducible set of second-class constraints, i.e. the matrix

$$C_{\Delta\Delta'} = [\tilde{\chi}_\Delta, \tilde{\chi}_{\Delta'}]\,, \tag{43}$$

is invertible, where $\tilde{\chi}_\Delta \equiv \left(\tilde{\chi}_{\alpha_{2k}}\right)_{k=\overline{0,\left[\frac{L}{2}\right]}}$.

The functions $A_{\alpha_{2k}}^{\alpha_{2k+1}}$ appearing in the above are defined by the relations:
-if L odd

$$\bar{A}_{\alpha_{2k}}^{\alpha_{2k+1}} = A_{\alpha_{2k}}^{\beta_{2k+1}} \hat{e}_{\beta_{2k+1}}^{\alpha_{2k+1}}, \qquad k = \overline{0, \left[\frac{L}{2}\right] - 1}, \tag{44}$$

$$\bar{A}_{\alpha_{L-1}}^{\alpha_L} = A_{\alpha_{L-1}}^{\beta_L} \bar{D}_{\beta_L}^{\alpha_L}; \tag{45}$$

-if L even

$$\bar{A}_{\alpha_{2k}}^{\alpha_{2k+1}} = A_{\alpha_{2k}}^{\beta_{2k+1}} \hat{e}_{\beta_{2k+1}}^{\alpha_{2k+1}}, \qquad k = \overline{0, \frac{L}{2} - 1}. \tag{46}$$

The elements $\hat{e}_{\beta_{2k+1}}^{\alpha_{2k+1}}$ determine an invertible matrix and iar $\bar{D}_{\beta_L}^{\alpha_L}$ are the elements of the inverse of the matrix of elements $D_{\alpha_L}^{\beta_L} = Z_{\alpha_L}^{\gamma_{L-1}} A_{\gamma_{L-1}}^{\beta_L}$.

The Dirac bracket built with respect to the irreducible second-class constraints (37) and (38) (or (39)–(41))

$$\begin{aligned}
&\left.[F,G]^{(L)*}\right|_{\text{ired}} = [F,G] - [F,\tilde{\chi}_{\alpha_0}]\,\mu^{(L)\alpha_0\beta_0}\,[\tilde{\chi}_{\beta_0},G] \\
&- \sum_{k=0}^{\left[\frac{L}{2}\right]-1} \Big\{ [F,\tilde{\chi}_{\alpha_{2k}}]\, Z_{\alpha_{2k+1}}^{\alpha_{2k}} \hat{e}_{\gamma_{2k+1}}^{\alpha_{2k+1}} \omega^{\gamma_{2k+1}\beta_{2k+1}} \bar{A}_{\beta_{2k+1}}^{\beta_{2k+2}} \left[\tilde{\chi}_{\beta_{2k+2}},G\right] \\
&+ \left[F,\tilde{\chi}_{\alpha_{2k+2}}\right] \bar{A}_{\alpha_{2k+1}}^{\alpha_{2k+2}} \omega^{\alpha_{2k+1}\gamma_{2k+1}} \hat{e}_{\gamma_{2k+1}}^{\beta_{2k+1}} Z_{\beta_{2k+1}}^{\beta_{2k}} \left[\tilde{\chi}_{\beta_{2k}},G\right] \\
&+ \left[F,\tilde{\chi}_{\alpha_{2k+2}}\right] \psi^{\alpha_{2k+2}\beta_{2k+2}} \left[\tilde{\chi}_{\beta_{2k+2}},G\right] \Big\}\,.
\end{aligned} \tag{47}$$

Teorema 6 *The Dirac bracket with respect to the irreducible second-class constraints (47) coincides with that of the intermediate system*

$$\left.[F,G]^{(L)*}\right|_{\text{ired}} \approx \left.[F,G]^{(L)*}\right|_{z,y}. \tag{48}$$

Based on (36) and (48), we are led to the relation

$$[F,G]^{(L)*} \approx \left.[F,G]^{(L)*}\right|_{\text{ired}}, \tag{49}$$

which expresses the fact that second-class constraints reducible of an arbitrary order L can be systematically approached in an irreducible manner.

The main results of the thesis are published in the papers:

1. C. Bizdadea, E. M. Cioroianu, S. O. Saliu, S. C. Sararu, **O. Balus**, J. Phys. A: Math Theor. 40 (2007) 14537

2. **O. Balus**, C. Bizdadea, E. M. Cioroianu, S. C. Sararu, Proceeding-ul conferintei "Physics Conference TIM-06" Timisoara, November 24-25, 2006, Analele Universitatii de Vest din Timisoara, 48 (2006) 58, Seria Fizica

3. C. Bizdadea, E. M. Cioroianu, S. C. Sararu, **O. Balus**, Irreducible analysis of reducible second-class constraints: the example of gauge-fixed three- and two-forms with Stueckelberg coupling, Rom. Rept. Phys. 62 (2010) 3

4. **O. Balus**, C. C. Ciobirca, D. Cornea, E. Diaconu, I. Negru, S. C. Sararu, Annals of the University of Craiova, Physics AUC 17 (part II) (2007) 51

5. C. Bizdadea, E. M. Cioroianu, I. Negru, S. O. Saliu, S. C. Sararu, **O. Balus**, Nucl. Phys. B812 (2009) 12

6. **O. Balus**, C. Bizdadea, E. M. Cioroianu, S. O. Saliu, S. C. Sararu, Proceeding-ul conferintei "Physics Conference TIM-07" Timisoara, November 23-24, 2007, Analele Universitatii de Vest din Timisoara, 50 (2007) 98, Seria Fizica

7. C. Bizdadea, **O. Balus**, E. M. Cioroianu, S. O. Saliu, S. C. Sararu, Proceedings of the "6th International Spring School and workshop on Quantum Field Theory and Hamiltonian Systems", 6-11 May 2008, Calimanesti-Caciulata, Romania, Annals of the University of Craiova, Physics AUC 18 (2008) 207

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9. E. M. Cioroianu, S. C. Sararu, **O. Balus**, First-class approaches to massive abelian 2-forms, Int. J. Mod. Phys. A25 (2010) 185