UNIVERSITY OF CRAIOVA
FACULTY OF PHYSICS

SUMMARY OF PH.D. THESIS

DYNAMICS OF NON-LINEAR PHYSICAL SYSTEMS

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I. Thesis Contents:

II. The importance of the thesis:

Dynamical systems have been extensively studied and present interest both within the fundamental theoretical research and applications in mathematics and physics. It is well known that most natural systems are nonlinear, which makes that their mathematical modeling is doing with nonlinear systems of equations. In nonlinear systems, due to the lack of the superposition principle, the general methods of solving missing or those that exist are applied to well defined classes of problems described by nonlinear equations. Moreover, the existence of non-linearity generates a much higher set of solutions and associated phenomena, some of which are essential for the understanding of natural phenomena.

The problem of nonlinear models is related to the study of the direct or indirect methods of integrability investigation, numerical solving of evolution equations and graphical modeling of phase-space for non-integrable systems, particularly for systems with chaotic behavior and methods of chaos control.

An important method of integrability research for nonlinear systems with a finite number of degrees of freedom is the study of their invariance to a 1-parametric group of point transformations. Lie symmetries, which leave invariant the equations of evolution of the system or Noether symmetries and the action of the system, are in the focus of theoretical research. Lately, there are searches for other types of symmetries. In the literature, such as for fast diffusion models, there were outlined new classes of solutions in 1-D space for nonlinear differential equation which describes this process. There were found non-local symmetries for the Kepler problem. The study was completed, proving that these symmetries can be calculated by standard methods of Lie theory, if the technique of order reduction is used. Another direction of research is the study of perturbation of symmetries of physical systems and the generation of associated adiabatic invariants with an important role in research of quasi-integrability of mechanical systems. Also building conservation laws for different nonlinear models is useful for the determination of conserved quantities, for the analysis of solutions and for the linearization of the system.

III. Structure of the thesis

Chapter 1 includes four sections. In the first two sections there are presented successively: a characterization of regular dynamics for integrable Hamiltonian systems; definition of the concept of chaos for nonlinear dynamical systems and two algorithmic methods of chaos control for perturbed Hamiltonian systems. Among the two methods of chaos control, one was introduced by M. Vittot and the second is an original contribution. The last two sections of the first chapter are designed for applications.

Firstly, we analyze the dynamic of the mechanical Yang-Mills model. In this context there will be emphasized different classes of quasi-periodical solutions and will be constructed the Poincaré sections for two Yang-Mills-type systems using numerical methods. An interesting connection between the Yang-Mills system and the famous Hénon-Héiles mechanical model is achieved by applying a "Swirl" invertible transformation on the first model.

Another application introduced in the first chapter is the implementation of the new algorithm of chaos control for two Hamiltonian systems with two degrees of freedom and polynomial potentials. Thus, we determine the chaos control functions for which the two nonlinear models become integrable. Unlike Vittot’s algorithm, we were able to calculate the "deformation" of the second invariant of the initial integrable system for each model, so that the controlled system becomes integrable.

Chapter 2 is structured in three sections. The first section analyzes a direct method for determining second invariants with various orders in velocities for 2D autonomous Hamiltonian systems. Applying this general algorithm for the dynamical models Yang-Mills and Hénon-Heiles is the objective of the second section. All cases of integrability and the concrete forms of the constants of motion obtained in this application are original results. In the last section of this chapter it is presented an algorithmic method for investigation of the invariants for the general case of 2D non-autonomous Hamiltonian systems.

Chapter 3 consists of three sections. The first two sections introduce successively: the construction method of symmetry generators for an arbitrary dynamical system, the general algorithm for the investigation of the Lie symmetry operators and determination of their associated invariants for autonomous Hamiltonian systems, respectively non-autonomous ones. The third section of this chapter explains in its first part the conditions of existence of variational symmetries for a system of differential equations resulting from a variational principle and also the construction method of conservation laws for such variational systems. Taking into account the difficulty of obtaining the variational symmetries and coordinate dependence of the Noether theorem that generates conservation laws, in the last part it is presented a general, direct method to obtain conservation laws for a nonlinear dynamical system, which may not be a variational system.

Chapter 4 has an applicative character. In its four sections there will be analyzed separately three physical models using general algorithmic methods introduced in the previous chapter.

The first model is the general autonomous Yang-Mills system with 2 degrees of freedom. For it there have already been determined the cases of integrability and the concrete forms of the invariants using the direct method presented in Chapter 2, and for its two particular cases there were found "traces of integrability" depending on initial conditions and has been studied the behavior of systems at different energies. The methodological study of the Lie symmetries and their invariants by the indirect method has been accepted for publication in the ISI journal IJTP.

The second application is done for the two-dimensional model of the Ricci flow, with applications in gravity (for example, to study black holes), in studying the Carleman model of Boltzmann equation, in plasma physics. The Lie symmetry operators and associated invariants, the non-classical operators and associated invariants obtained for the two-dimensional model and the generation of the density of conservation quantity for the 1-dimensional model, are not published by other authors.

A third application is the nonlinear heat equation. Using algorithmic method for investigation of the classical symmetries there will be achieved the differential system which will determine the Lie operators for the general form of nonlinear heat equation. The general system will be solved for 2 particular cases. There will be generated the Lie symmetry operators for these cases, then there will be identified the associated invariants and some similarity solutions for the analyzed models will be pointed out. There will be highlighted the

connections between the 2-dimensional problem studied using 1-dimensional problem described in literature with the results obtained by the author for 2D Ricci flow model and outcomes published in ISI journal JNMP.
The main results will be unified in the concluding chapter.
Numerical programs used to obtain various graphic representations will be included in the Appendix.

IV. Main conclusions

The main conclusions of the study of integrability, chaos, or symmetries of nonlinear dynamical systems with finite number of degrees of freedom, achieved in this work, can be summarized as follows.

In the first chapter, there have been highlighted "traces of integrability" for two mechanical models of Yang-Mills type with two degrees of freedom associated with Yang-Mills field. More exactly, there have been highlighted, using numerical methods, classes of quasi-periodical solutions with sensitive dependence on the set of initial conditions. Moreover, this dependence is specific to chaotic dynamical systems, systems from which Yang-Mills field is part. By numerical analysis there were constructed Poincaré surfaces of the section $x = 0, \quad y = 0$ for the above mentioned models. In Section 1.3 it was found, for one particular case of Yang-Mills mechanical model, a regular behavior at a low energy of system and an evolution to chaos with increasing energy. For the second case it was observed a more stable behavior, the emergence of chaos being realized at a higher energy of the system.

We have also identified a transformation of "Swirl" type, which applied to the Yang-Mills model leads to a phase portrait similar to the well known Hénon-Heiles system, for sensible close values of the energies of the two systems. In this chapter, in subsection 1.2.2 we proposed an original new algorithm for chaos control of Hamiltonian perturbed systems. The algorithm allows finding a control function of the same order ($O(\varepsilon)$) with the applied perturbation of the initial chosen integrable Hamiltonian system. This is in opposition to Vittot's algorithm, where the control term is calculated to order $O(\varepsilon^2)$ in relation to the fluctuation of the regulate system. In addition to "Vittot's method", the new algorithm succeeds to identify the perturbation which must "deform" the second invariant of the initial integrated system so that the controlled chaotic system will admit a more regular dynamics.

This possibility was tested for two Hamiltonian models with polynomial potentials. Proof of the chaos control for the cited examples was supported by the construction of Poincaré sections.

In Chapter 2 it is presented a direct method for construction of invariants for autonomous and non-autonomous Hamiltonian systems with 2 degrees of freedom. This algorithmic method requires the search of a particular form of invariants (for example, with different orders in velocities). For application section, there were chosen two 2-D general autonomous models: a) the Yang-Mills system, for which it has been identified a case of integrability in which the second invariant (the first invariant is the Hamiltonian itself) is linear in velocities (angular momentum), and three cases, in which the second constant of motion is quadratic in velocities; b) the Hénon-Heiles system, which admits two integrable forms for which second invariants are quadratic in velocities, and a single case in which the second constant of motion is quartic in velocities.

The 3rd chapter approaches the indirect method and determines the Lie invariants for nonlinear dynamical systems. Specifically, it presents an algorithm for determining the invariants, in conditions that were previously identified the Lie symmetry operators that leave invariant the equations of evolution of the studied systems. Also, a direct method for construction of conservation laws for dynamical systems, variational or not, was exposed at the end of this chapter.

Chapter 4 includes examples of algorithms for investigating the Lie symmetries generators and associated invariants for 3 different physical models:

1) The general autonomous mechanical Yang-Mills model, with 2 degrees of freedom: Firstly, there have been determined the concrete forms of four Lie symmetry operators with coefficient functions that are linear in velocities. Then, from the condition of existence of associated invariant with a generator of symmetry, there were identified the second invariants and 4 cases of integrability of the model. The results of this indirect algorithm coincide with those obtained by the direct method presented in Chapter 2.

2) The two-dimensional Ricci flow model, with applications in gravity:

For this model there has been identified a symmetry operator whose expression depends on 2 parameters and 2 arbitrary functions. Choosing linear forms for arbitrary functions one can generate a set of 6 Lie symmetry operators. These operators satisfy a Lie algebra, which can be decomposed into 3 independent subalgebras admitting an interesting matrix representation. The forms of these operators generate the symmetry of the equation of evolution at time and space translations, on the scale transformation and dilatation. Lie symmetry invariants generated by the symmetry operators have interesting shapes, from the simplest (coordinates itself) to arbitrary functions. Using some invariants for imposing the similarity conditions, there was obtained a very simple solution to the equation of motion of the 2D Ricci model as stationary solutions or a solution that propagates linearly in time. It is exposed the general problem of non-classical symmetries for the 2D model of Ricci flow. Solving this problem led to the rediscovery of the Lie symmetry operators (classical operators) and also highlighted 4 cases of non-classical symmetry generators. There have also been determined the non-classical invariants associated with non-classical symmetry operators. For the 1-dimensional model of the Ricci flow, or equivalent for fast diffusion equation, there was built, under the general method described in Chapter 3, subsection 3.3.2, the law of conservation, which determined the conserved current density. In a future work, the author of the thesis intends to determine the density of the conservative quantities for the 2D-Ricci model, which involves more complex calculations, and other models of physical interest.

3) Nonlinear heat equation

For the two-dimensional model of nonlinear heat equation, using the classical algorithm, a general differential system has been obtained for the function $f(t)$ which expresses the non-linearity of the model and for the coefficient functions $\xi^1(x,t), \xi^2(y,t), \varphi(t), \phi(x,y,t,u)$ of the general Lie symmetry. The Lie symmetry generators and their associated invariants have been found in the cases: $f(t)=u^{\alpha}$, $f(t)=e^{u}$. Using the expressions of the invariants there were highlighted interesting similarity solutions for the studied model. For the case $f(t)=u^{\alpha}$ there were compared the results for the 2D analyzed model with those already known for the 1D model. For the 2D model with the choice $f(t) = u^{\alpha}, \quad \alpha = -1$, there were rediscovered the Lie operators for the Ricci model obtained in the Section 4.2.

The author wishes to continue the study of symmetries for nonlinear dynamical systems by extending the investigation spectrum. Her future interest will be to study other relevant physical models and analyze the existence of other types of symmetries, not necessarily Lie (classical symmetries). A first step was achieved. The author was able to identify the constraints that must be imposed on a dynamical system in order to admit a certain type of symmetry (e.g. Galilei-type symmetry).

V. Selected References

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